On Generalized Distributions and Pathways

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Abstract. The scalar version of the pathway model of Mathai (2005) is
shown to be associated with a large number of probability models used in
physics. Different families of densities are listed here, which are all connected
through the pathway parameter α, generating a distributional pathway. The
idea is to switch from one functional form to another through this parameter and it is shown that one can proceed from the generalized type-1 beta
family to generalized type-2 beta family to generalized gamma family. It is
also shown that the pathway model is available by maximizing a generalized
measure of entropy, leading to an entropic pathway, covering the particularly
interesting cases of Tsallis statistics (Tsallis, 1988) and superstatistics (Beck
and Cohen, 2003).

1 Introduction

When deriving or fitting models for data from physical experiments very
often the practice is to take a member from a parametric family of distributions. But it is often found that the model requires a distribution with
a more specific tail than the ones available from the parametric family, or
a situation of right tail cut-off. The model may reveal that the underlying
distribution is in between two parametric families of distributions. In order
to create a distributional pathway for proceeding from one functional form
to another a pathway parameter α is introduced and a pathway model is created in Mathai (2005). Section 1 defines this pathway model. In Section 2
it is shown that this model enables one to go from a generalized type-1 beta
model to a generalized type-2 beta model to a generalized gamma model
when the variable is restricted to be positive. More families are available
when the variable is allowed to vary over the real line. Corresponding mul-

titudes of families are available when the variable is in the complex domain. This is a pragmatic approach to the distributional pathway. Mathai (2005) deals mainly with rectangular matrix-variate distributions in the real case and the scalar case is the topic of discussion in the current paper because the scalar case is connected to many problems in physics. Further, in Section 3, through a generalized measure of entropy, an entropic pathway can be devised that leads to different families of entropic functionals that produce families of distributions by applying the maximum entropy principle (Mathai and Haubold, 2007). This is a theoretical approach to the distributional pathway. Section 4 draws conclusions from the development of the distributional pathway.

2 The pathway model in the real scalar case

For the real scalar case the pathway model for positive random variables is represented by the following probability density, for purely mathematical statistical reasons, and to unify and extend the compilation of statistical distributions in Mathai (1993):

$$f(x) = c \, x^{\gamma-1}[1 - a(1-\alpha)x^\delta]^{\frac{\beta}{1-\alpha}}, \quad x > 0, \tag{1}$$

$a > 0, \delta > 0, \beta \geq 0, 1 - a(1-\alpha)x^\delta > 0, \gamma > 0$ where c is the normalizing constant and α is the pathway parameter. The normalizing constant in this case is the following:

$$c = \frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}, \quad \text{for } \alpha < 1 \tag{2}$$

$$= \frac{\delta[a(\alpha-1)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right)}, \quad \text{for } \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0, \ \alpha > 1 \tag{3}$$

$$= \frac{\delta\,(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \quad \text{for } \alpha \to 1. \tag{4}$$

We note here with respect to (1) that in information theory and mathematical statistics, parameter interpretation is less important than in special cases under the interpretation of a physical model or principle (Kaniadakis and Lissia, 2004).

2.1 Generalized type-1 beta family emanating from equation (1)

For $\alpha < 1$ or $-\infty < \alpha < 1$ the model in (1) remains as a generalized type-1 beta model in the real case, where the right tail is cut-off. In a series of publications, see for example Mathai and Haubold (1988) and Haubold and Mathai (2000), we considered various modifications to the Maxwell-Boltzmann approach to the nuclear reaction rate theory and considered the situations of nonresonant reactions and cases of depleted tail and tail cut-off. It may be observed that all those situations are covered by the pathway model in (1). Before proceeding with the properties let us look at some special cases first. For $\alpha = 0$, $a = 1$, $\delta = 1$ with β replaced by $\beta - 1$ we have the regular type-1 beta density, namely,

$$f_1(x) = c_1 \; x^{\gamma-1}(1-x)^{\beta-1}, 0 < x < 1, \tag{5}$$

where $c_1 = \frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)\Gamma(\beta)}$. For $a = 1, \gamma = 1, \delta = 1, \beta = 1$ we have Tsallis statistics (Tsallis, 1988; Gell-Mann and Tsallis, 2004) for $\alpha < 1$, namely,

$$f_2(x) = c_2 \; [1 - (1-\alpha)x]^{\frac{1}{1-\alpha}}, \;\; 0 < x < (1-\alpha)^{-1}, \tag{6}$$

where c_1 is the corresponding normalizing constant. For $\alpha = 0, a = 1$ in (1) we have the power function densities for $\gamma = 1$ and $\beta = 0$ respectively, namely,

$$f_3(x) = c_3 \; x^{\gamma-1}, \;\; 0 < x < 1 \tag{7}$$

and

$$f_4(x) = c_4 \; (1 - x^\delta)^\beta, \; 0 < x < 1, \tag{8}$$

where c_3, c_4 are the corresponding normalizing constants. Note that Pareto densities (Brouers and Sotolongo-Costa, 2004; Shalizi, 2007) also come from the power function models. Further, if $\gamma = 1$, $\delta = 1$, $a = 1, \alpha = 0$, $\beta = 0$ in (1) we have the uniform density,

$$f_5(x) = 1, \; 0 < x < 1. \tag{9}$$

Before concluding this section a remark on Bose-Einstein statistics (Ijiri and Simon, 1975; Aragao-Rego et al. 2003) in physics is in order. Naturally one would expect the Bose-Einstein density to belong to the type-1 beta family.

If we take the parameters $\gamma = 1$ and $\beta = 0$ in the density in equation (5) then naturally the function goes to

$$g(x) = c^* \frac{1}{1-x}$$

and obviously $g(x)$ cannot make it a density in $0 \leq x \leq 1$ unless x is bounded away from 1. This can be achieved by a transformation of the type $x = \exp(-t - wy)$ for $w > 0, e^t > 1$. In this case $g(x)$ goes to

$$g_1(y) = c_1^* \frac{w}{e^{t+wy} - 1}, \quad 0 \leq y < \infty, \ w > 0, \ e^t > 1. \tag{10}$$

This is nothing but the Bose-Einstein density which is given by

$$f_6(y) = c_6 \frac{1}{-1 + \exp(t + wy)}, \quad w > 0, 0 \leq y < \infty, \ e^t > 1. \tag{11}$$

Remark 2.1 Thus Bose-Einstein density is a limiting form of a type-1 beta density of equation (5) where the variable x is transformed by the above inverse transformation where the normalizing constant c_6 cannot be obtained through gamma functions but it can be evaluated by appealing to partial fractions technique and then taking logarithms and it can be easily seen to be the following:

$$c_6 = w \left[\ln\left(\frac{e^t}{e^t - 1} \right) \right]^{-1}.$$

2.2 Generalized type-2 beta family emanating from equation (1)

When $\alpha > 1$ in (1) we may write $1 - \alpha = -(\alpha - 1), \alpha > 1$ so that $f(x)$ assumes the form,

$$f(x) = c \, x^{\gamma-1}[1 + a(\alpha - 1)x^\delta]^{-\frac{\beta}{\alpha-1}}, \quad x > 0 \tag{12}$$

which is a generalized type-2 beta model for real x. Beck and Cohen's superstatistics (Beck and Cohen, 2003; Beck 2006) belongs to this case (12). For $\alpha = 2, a = 1, \delta = 1, \beta - \gamma > 0$ we have the regular type-2 beta density, namely,

$$f_7(x) = c_7 \, x^{\gamma-1}(1 + x)^{-\beta}, \quad x > 0. \tag{13}$$

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In $f_7(x)$ for $x = \frac{m}{n}F, \ m, n = 1, 2, \ldots, \gamma = \frac{m}{2}, \beta = \frac{m}{2} + \frac{n}{2}$ we have the F density or the variance-ratio density, namely,

$$f_8(F) = c_8 \ F^{\frac{m}{2}-1}(1 + \frac{m}{n}F)^{-(\frac{m}{2}+\frac{n}{2})}, \ \ F > 0. \tag{14}$$

For $\gamma = 1, a = 1, \delta = 1, \ \beta = 1$ in (12) we have Tsallis statistics (Tsallis, 1988; Gell-Mann and Tsallis, 2004) for $\alpha > 1$, namely

$$f_9(x) = c_9 \ [1 + (\alpha - 1)x]^{-\frac{1}{\alpha-1}}, \ x > 0. \tag{15}$$

For $\delta = 2, -\infty < x < \infty, a = \frac{1}{\nu}, \alpha = 2, \beta = \frac{\nu+1}{2}$ in (12) we have the Student-t density (Gheorghiu and Coppens, 2004), namely

$$f_{10}(x) = c_{10}(1 + \frac{x^2}{\nu})^{-\frac{(\nu+1)}{2}}, \ \ -\infty < x < \infty. \tag{16}$$

In f_{10} for $\nu = 1$ we have the Cauchy model, namely,

$$f_{11}(x) = c_{11}(1 + x^2)^{-1}, \ \ -\infty < x < \infty. \tag{17}$$

In f_7 if $\gamma = 1$, $\beta = 1$, $x = e^{\epsilon+\eta y}$ for $\eta > 0$, $\epsilon \neq 0$, $0 \leq y < \infty$ then we have the Fermi-Dirac density (Aragao-Rego et al., 2003)

$$f_{12}(y) = c_{12}[1 + \exp{(\epsilon + \eta y)}]^{-1}, 0 \leq y < \infty. \tag{18}$$

Remark 2.2 Observe that f_{12} is a limiting form of the ordinary type-2 beta model with $\beta - \gamma = 0$ in f_7. In this case the normalizing constant cannot be evaluated with the help of gamma functions but c_7 can be evaluated through partial fractions and then appealing to logarithms.
In f_7 if we transform x to y such that $x = e^y$, $-\infty < y < \infty$ then we obtain the generalized logistic and related models

$$f_{13}(y) = c_{13}[e^y]^\gamma[1 + e^y]^{-\beta}, -\infty < y < \infty \tag{19}$$

which are applicable in many areas of statistical analysis, see for example Mathai and Provost (2006). Before concluding this section one more remark is in order.
Remark 2.3 For the generalized type-2 beta model, that is the model in (1) or the model in (12) for $\alpha > 1$, x and $\frac{1}{x}$ belong to the same family of distributions. Hence we could have replaced x^δ by $x^{-\delta}$ in the densities in equations (13) to (19) then we would obtain the corresponding additional families of densities.

2.3 Generalized gamma family emanating from equation (1)

When $\alpha \to 1$ the forms in (1) for $\alpha < 1$ and for $\alpha > 1$ reduces to

$$f(x) = c\, x^{\gamma-1}e^{-a\beta\, x^\delta}, \quad x > 0. \tag{20}$$

This includes generalized gamma, gamma, exponential, chisquare, Weibull, Maxwell-Boltzmann, Rayleigh, and related models (Mathai, 1993). The model in equation (20) is generally known as the generalized gamma model. For $\gamma = 1, \delta = 2, -\infty < x < \infty, a = 1$, in (20) we have the Gaussian density or error curve,

$$f_{14}(x) = c_{14}\exp\left(-\beta x^2\right), -\infty < x < \infty. \tag{21}$$

For $\gamma = \delta, a = 1$ in (20) we have the Weibull density, given by

$$f_{15}(x) = c_{15}x^{\delta-1}e^{-\beta x^\delta}, \quad x > 0. \tag{22}$$

For $\delta = 1, a = 1$ in (20) we have the gamma density, given by

$$f_{16}(x) = c_{16}x^{\gamma-1}e^{-\beta x}, \quad x > 0. \tag{23}$$

For $\gamma = \frac{n}{2}$ and $\beta = \frac{1}{2}$ in f_{16} we have the chisquare density with n degrees of freedom, given by

$$f_{17}(x) = c_{17}x^{\frac{n}{2}-1}e^{-x/2}, \quad x > 0. \tag{24}$$

For $\gamma = n, \beta = \frac{n}{2\sigma^2}, \sigma > 0, n = 1, 2, ..., \delta = 2, a = 1$ in (20) we have the chi density, given by

$$f_{18}(x) = c_{18}x^{n-1}e^{-\frac{nx^2}{2\sigma^2}}, \quad x > 0. \tag{25}$$

For $\gamma = 2, \ldots$ in f_{16} we have the Erlang density, given by

$$f_{19} = c_{19}x^{p-1}e^{-\beta x}, p = 2, \ldots, \quad x > 0. \tag{26}$$

For $p = 1$ in f_{19} we have the exponential density, given by

$$f_{20}(x) = c_{20}e^{-\beta x}, \quad x > 0. \tag{27}$$

For $\delta = 2, \gamma = 3, a = 1$ in (20) we have the Maxwell-Boltzmann density, given by

$$f_{21}(x) = c_{21}x^2e^{-\beta x^2}, \quad x > 0. \tag{28}$$

For $\delta = 2, \gamma = 2, \beta = \frac{1}{2\sigma^2}, \sigma > 0, a = 1$ in (20) we have the Rayleigh density (Tirnakli et al., 1998), given by

$$f_{22}(x) = c_{22} x e^{-\frac{x^2}{2\sigma^2}}, x > 0. \tag{29}$$

Remark 2.4 Observe that in (20) if we replace $\delta > 0$ by $-\delta, \delta > 0$ still the family belongs to the generalized gamma family. Hence a sequence of densities are available by replacing x by $\frac{1}{x}$ in (20).

Remark 2.5 If x is replaced by $|x|$ for $-\infty < x < \infty$ in the models (1), (12) and (20) we obtain a series of other densities which will also include double exponential or the Laplace density, among others.

Remark 2.6 In many practical problems there may be a location parameter or the variable may be located at a point different from zero. All such cases can be covered by replacing x in the pathway model in (1) by $(x - b)$ when $x - b > 0$ or by $|(x - b)|$ when $-\infty < x < \infty$, where b is a constant, called the location parameter.

Observe that in (12) and (20), $\frac{1}{x}$ also belongs to the same family of densities and hence in (12) and (20) one could have also taken $x^{-\delta}$ with $\delta > 0$.

3   Pathway model from a generalized entropy measure

A generalized entropy measure of order α, explored by Mathai and Haubold (2007), is a generalization of Shannon entropy and it is a variant of the generalized entropy of order α in Mathai and Rathie (1975). In the discrete case the measure is the following: Consider a multinomial population $P = (p_1, \ldots, p_k)$, $p_i \geq 0$, $i = 1, \ldots, k$, $p_1 + \ldots + p_k = 1$. Define the function

$$M_{k,\alpha}(P) = \frac{\sum_{i=1}^{k} p_i^{2-\alpha} - 1}{\alpha - 1}, \ \alpha \neq 1, \ -\infty < \alpha < 2 \tag{30}$$

$$\lim_{\alpha \to 1} M_{k,\alpha}(P) = -\sum_{i=1}^{k} p_i \ln p_i = S_k(P) \tag{31}$$

by using L'Hospital's rule. In this notation $0 \ln 0$ is taken as zero when any $p_i = 0$. Thus (30) is a generalization of Shannon entropy $S_k(P)$ as seen from (31). Note that (30) is a variant of Harvda-Charvát entropy $H_{k,\alpha}(P)$ and Tsallis entropy $T_{k,\alpha}(P)$ where

$$H_{k,\alpha}(P) = \frac{\sum_{i=1}^{k} p_i^{\alpha} - 1}{2^{1-\alpha} - 1}, \ \alpha \neq 1, \ \alpha > 0 \tag{32}$$

and

$$T_{k,\alpha}(P) = \frac{\sum_{1=1}^{k} p_i^{\alpha} - 1}{1 - \alpha}, \ \ \alpha \neq 1, \ \ \alpha > 0. \tag{33}$$

We will introduce another measure associated with (31) and parallel to Rényi entropy $R_{k,\alpha}$ in the following form:

$$M_{k,\alpha}^{*}(P) = \frac{\ln \left(\sum_{i=1}^{k} p_i^{2-\alpha} \right)}{\alpha - 1}, \ \ \alpha \neq 1, -\infty < \alpha < 2. \tag{34}$$

Rényi entropy is given by

$$R_{k,\alpha}(P) = \frac{\ln \left(\sum_{i=1}^{k} p_i^{\alpha} \right)}{1 - \alpha}, \ \ \alpha \neq 1, \ \alpha > 0. \tag{35}$$

It will be seen later that the form in (30) is amenable to the pathway model.

3.1   Continuous analogue

The continuous analogue to the measure in (30) is the following:

$$\begin{aligned} M_{\alpha}(f) &= \frac{\int_{-\infty}^{\infty} [f(x)]^{2-\alpha} dx - 1}{\alpha - 1} \tag{36} \\ &= \frac{\int_{-\infty}^{\infty} [f(x)]^{1-\alpha} f(x) dx - 1}{\alpha - 1} = \frac{E[f(x)]^{1-\alpha} - 1}{\alpha - 1}, \ \ \alpha \neq 1, \ \alpha < 2, \end{aligned}$$

where $E[\cdot]$ denotes the expected value of $[\cdot]$. Note that when $\alpha = 1$, $E[f(x)]^{1-\alpha} = E[f(x)]^0 = 1$.

It is easy to see that the generalized entropy measure in (30) is connected to Kerridge's "inaccuracy" measure (Kerridge, 1961). The generalized inaccuracy measure is $E[q(x)]^{1-\alpha}$ where the experimenter has assigned $q(x)$

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for the true density $f(x)$, where $q(x)$ could be an estimate of $f(x)$ or $q(x)$ could be coming from observations. Through disturbance or distortion if the experimenter assigns $[f(x)]^{1-\alpha}$ for $[q(x)]^{1-\alpha}$ then the inaccuracy measure is $M_\alpha(f)$ of (36).

3.2 Distributions with maximum generalized entropy

Among all densities, which one will give a maximum value for $M_\alpha(f)$ in equation (36)? Consider all possible functions $f(x)$ such that $f(x) \geq 0$ for all x, $f(x) = 0$ outside (a, b), $a < b$, $f(a)$ is the same for all such $f(x)$, $f(b)$ is the same for all such f, $\int_a^b f(x)dx < \infty$. Let $f(x)$ be a continuous function of x with continuous derivatives in (a, b). Let us maximize $\int_a^b [f(x)]^{2-\alpha}dx$ for fixed α and over all functional f, under the conditions that the following two moment-like expressions be fixed quantities:

$$\int_a^b x^{(\gamma-1)(1-\alpha)} f(x)dx = \text{given, and } \int_a^b x^{(\gamma-1)(1-\alpha)+\delta} f(x)dx = \text{given} \quad (37)$$

for fixed $\gamma > 0$ and $\delta > 0$. Consider

$$U = [f(x)]^{2-\alpha} - \lambda_1 x^{(\gamma-1)(1-\alpha)} f(x) + \lambda_2 x^{(\gamma-1)(1-\alpha)+\delta} f(x), \ \alpha < 2, \ \alpha \neq 1 \quad (38)$$

where λ_1 and λ_2 are Lagrangian multipliers. Then the Euler equation is the following:

$$\frac{\partial U}{\partial f} = 0 \ \Rightarrow \ (2-\alpha)[f(x)]^{1-\alpha} - \lambda_1 x^{(\gamma-1)(1-\alpha)} + \lambda_2 x^{(\gamma-1)(1-\alpha)+\delta} = 0$$

$$\Rightarrow \ [f(x)]^{1-\alpha} = \frac{\lambda_1}{(2-\alpha)} x^{(\gamma-1)(1-\alpha)} [1 - \frac{\lambda_2}{\lambda_1} x^\delta] \quad (39)$$

$$\Rightarrow \ f(x) = c \, x^{\gamma-1} [1 - \eta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}} \quad (40)$$

where λ_1/λ_2 is written as $\eta(1-\alpha)$ with $\eta > 0$ such that $1 - \eta(1-\alpha)x^\delta > 0$ since $f(x)$ is assumed to be non-negative. By using the conditions (37) and (39) we can determine c and η. When the range of x for which $f(x)$ is nonzero is $(0, \infty)$ and when c is a normalizing constant, then (40) is the pathway model of Mathai (2005) in the scalar case where α is the pathway parameter. When $\gamma = 1, \delta = 1$ in (40) then (40) produces the power law. The form in (39) for various values of λ_1 and λ_2 can produce all the four forms

$$\alpha_1 x^{\gamma-1} [1 - \beta_1(1-\alpha)x^\delta]^{-\frac{1}{1-\alpha}}, \ \alpha_2 x^{\gamma-1} [1 - \beta_2(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}} \text{for} \alpha < 1$$

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and

$$\alpha_3 x^{\gamma-1}[1+\beta_3(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}, \ \alpha_4 x^{\gamma-1}[1+\beta_4(\alpha-1)x^\delta]^{\frac{1}{\alpha-1}} \text{for} \alpha>1$$

with $\alpha_i, \beta_i > 0, i = 1, 2, 3, 4$. But out of these, the second and the third forms can produce densities in $(0, \infty)$. The first and fourth will not be converging. When $f(x)$ is a density in (40), what is the normalizing constant c? We need to consider three cases of $\alpha < 1, \alpha > 1$ and $\alpha \to 1$. This c is already evaluated in section 1.

Remark 3.1 In Mathai and Haubold (2007) further results are provided for the pathway model associated with Tsallis' entropy, fractional calculus, Mittag-Leffler functions, and distributions with one of the parameters having a prior distribution of its own, giving rise to superstatistics (Beck and Cohen, 2003; Beck, 2006).

4   Conclusions

Based on the pathway model developed by Mathai (2005) we derived a *distributional pathway* proceeding from the generalized type-1 beta family to generalized type-2 beta family in eq. (12) to generalized gamma family in eq. (20). This distributional pathway encompasses, among others, the distributions of Maxwell-Boltzmann and Tsallis (q-exponential function, see Gell-Mann and Tsallis 2004; Tsallis, 2004) that are fundamental to Boltzmann-Gibbs statistical mechanics and its nonextensive generalization by Tsallis, respectively, as well as Bose-Einstein and Fermi-Dirac distributions. Subsequent to the distributional pathway, an *entropic pathway*, emanates from eq. (1) by maximum principle applied to the generalized entropic form of order α in (30), covering entropic functionals of Shannon, Boltzmann-Gibbs, Rényi, Tsallis, and Harvda-Charvát (Mathai and Haubold, 2007). The results in this paper are a contribution to the on-going debate in the physical literature concerning the generalization of Boltzmann-Gibbs statistical mechanics through a one-parameter generalization of Boltzmann-Gibbs entropy, known as Tsallis entropy contained in the seminal paper of Tsallis (1988), whose form is equivalent to that of Harvda-Charvát, and which opened the door to generalizations of Boltzmann-Gibbs statistical mechanics.

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